AD-A116 021 IOWA STATE UNIV AMES DEPT OF STATISTICS F/G 12/1
A GLOBAL EXISTENCE AND UNIQUENESS THEOREM FOR A RICCATI EQUATIO--ETC(U)
1981 J C KEGLEY AFOSR-78-3518
UNCLASSIFIED AFOSR-TR-82-0491 NL

END
DATE
FILMED
7-82
DTIC

CONT

OF

A

6021

AFOSR-TR- 82-0491

AD A116021

A GLOBAL EXISTENCE AND UNIQUENESS THEOREM FOR A RICCATI EQUATION*

J. COLBY KEGLEY[+]

Abstract. It is proved that a Riccati differential equation of a particular form has a unique solution satisfying the conditions that it is to exist for large values of the independent variable t and to have its graph stay above a certain line for large t. It is then proved that the solution exists for all t. Two forms of the solution are developed in terms of the confluent hypergeometric functions. An application of these results is made to an asymptotic stochastic analysis of a noisy duel problem.

DTIC
ELECTE
JUN 23 1982
S D
H

DTIC FILE COPY

*This [redacted] paper was partially supported by AFOSR Grant No. AFOSR-78-3518.

[+]Department of Mathematics, Ames, Iowa 50011.

Approved for public release;
distribution unlimited.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER AFOSR-TR- 82-0491	2. GOVT ACCESSION NO. AD-A116 021
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle) A GLOBAL EXISTENCE AND UNIQUENESS THEOREM FOR A RICCATI EQUATION	5. TYPE OF REPORT & PERIOD COVERED Interim [redacted] 1 DEC 77-31 AUG 81
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) J. Colby Kegley	8. CONTRACT OR GRANT NUMBER(s) AFOSR-78-3518
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Statistics, Iowa State University, Ames IA 50011	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS PE61102F; 2304/A5
11. CONTROLLING OFFICE NAME AND ADDRESS Mathematical & Information Sciences Directorate, Air Force Office of Scientific Research, Bolling AFB DC 20332	12. REPORT DATE 1981
	13. NUMBER OF PAGES 21
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report) UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Riccati differential equation.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

It is proved that a Riccati differential equation of a particular form has a unique solution satisfying the conditions that it is to exist for large values of the independent variable t and to have its graph stay above a certain line for large t. It is then proved that the solution exists for all t. Two forms of the solution are developed in terms of the confluent hypergeometric functions. An application of these results is made to an asymptotic stochastic analysis of a noisy duel problem.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

1. Introduction. This paper investigates a particular form (1) in section 2 of a Riccati equation that is quadratic in the independent variable t. The approach to the problem of the existence of a global solution is not from the usual initial-value standpoint, but is based on a desired feature of the solution for large t which is given by properties (i) and (ii) of the Theorem in section 2. The form (1) of the equation makes it easy to draw rough sketches of how the solutions behave depending on where the initial point is selected. In particular, it becomes plausible that there exists a solution defined for all t that satisfies properties (i) and (ii), but it is by no means clear that there is only one such solution. That this is the case indicates that this distinguished solution is extremely unstable. Indeed, one of the implications of Lemma 5 in section 3 is that every other solution diverges from the distinguished solution as $t \to \infty$.

Our investigation is motivated by the approach used in [3] and [6] to analyze the equal-accuracy noisy duel problem for two players having finite unequal units of ammunition. This approach leads to asymptotic distributions of normalized times of first fire for the two players. The hazard rates for these distributions are expressed in terms of a solution to a Riccati equation of the form (1), and the distributions themselves are expressed in terms of a solution to a related Hermite equation.

A brief outline of these connections is given in section 4. The reader may find it helpful to read that section in conjunction with the statement of the Theorem to understand the reason for deriving the various properties of the distinguished solution.

2. Statement of the Theorem. The principal conclusions we desire can be stated as follows.

AIR FORCE OFFICE OF SCIENTIFIC RESEARCH (AFSC)
NOTICE OF TRANSMITTAL TO DTIC
This technical report has been reviewed and is approved for public release IAW AFR 190-12.
Distribution is unlimited.
MATTHEW J. KERPER
Chief, Technical Information Division

THEOREM. Suppose α is a positive number and ϕ_1, ϕ_2 are linear functions $\phi_i(t) = \beta_i t + \gamma_i$, where $\beta_2 < \beta_1$ and $\beta_1 > 0$. Then there is exactly one solution of the Riccati equation

$$v'(t) = \alpha[v(t) - 2\phi_1(t)][v(t) - 2\phi_2(t)] \tag{i}$$

that has the following two properties: There is a number t_0 such that

(i) The domain of v includes the interval $[t_0,\infty)$;

(ii) $v(t) - 2\phi_1(t) > 0$ for $t \geq t_0$.

Moreover, this solution has the additional properties:

(iii) The domain of v is $(-\infty,\infty)$.

(iv) $v(t) - 2\phi_1(t) > 0$ for all t.

(v) $\int^{\infty}[v(t) - 2\phi_1(t)]dt = \int^{\infty}[v(t) - 2\phi_2(t)]dt = \infty$.

(vi) $v(t) - 2\phi_1(t) \to 0$ as $t \to \infty$.

(vii) $v'(t) \to 2\beta_1$ as $t \to \infty$.

If in addition, $\beta_2 < 0$, then the following hold:

(viii) If t_0 is a number, then the conditions

$$v(t) - 2\phi_2(t) > 0 \quad \text{for} \quad t > t_0$$

and $v(t_0) - 2\phi_2(t_0) = 0$

hold exactly when

$$\phi_1(t_0) - \phi_2(t_0) = z_0\sqrt{(\beta_1-\beta_2)/2\alpha},$$

where z_0 is the real zero of the Weber parabolic cylinder function D_{p+1} with $p = \beta_1/(\beta_2-\beta_1)$.

(ix) With t_0 as in (viii) and, for $i = 1,2$, we define $f_i(t) = x_i(t)\bar{\Phi}_i(t)$, where $x_i(t) = \alpha[v(t) - 2\phi_i(t)]$ and $\bar{\Phi}_i(t) = \exp(-\int_{t_0}^{t} x_i(\tau)d\tau)$, we have:

(ixa) $\bar{\Phi}_1(t) < \bar{\Phi}_2(t)$ when $t_0 < t < -t_0 - 2\eta$ and

$\bar{\Phi}_1(t) > \bar{\Phi}_2(t)$ when $t > -t_0 - 2\eta$,

where $t = -\eta$ is the solution of $x_1(t) = x_2(t)$.

Accession For
NTIS GRA&I ☑
DTIC TAB ☐
Unannounced ☐
Justification
By
Distribution/
Availability Codes
Avail and/or
Dist Special
A

DTIC COPY INSPECTED 2

(ixb) f_1 is decreasing and positive on $(-\infty,\infty)$.

(ixc) f_2 is positive on (t_0,∞) and has a maximum value that occurs at a number $t_1 > -\eta > t_0$.

(ixd) $\int_{t_0}^{\infty} tf_2(t)dt < \infty$

(ixe) $\int_{t_0}^{\infty} tf_1(t)dt = \infty$

In the process of proving this theorem, two forms of the solutions are developed.

(A) $$v(t) = 2\phi_1(t) - \frac{\delta}{\alpha}\frac{\psi'(s)}{\psi(s)} ,$$

where $s = \delta(t+\eta)$, $\delta = \sqrt{\alpha(\beta_1-\beta_2)}$, $\eta = (\gamma_1-\gamma_2)/(\beta_1-\beta_2)$,

and $$\psi(s) = \zeta\, y_0(s) + y_1(s),$$

with: $\zeta = -2\Gamma(a + 1/2)/\Gamma(a)$, $a = \beta_1/2(\beta_1-\beta_2)$,

$$y_0(s) = s\ {}_1F_1(a + 1/2,\ 3/2;\ s^2),$$

$$y_1(s) = {}_1F_1(a,\ 1/2;\ s^2),$$

and ${}_1F_1$ denotes the confluent hypergeometric function

$${}_1F_1(a,\ b;\ z) = \frac{\Gamma(b)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(b+n)} \frac{z^n}{n!} .$$

(B) $$v(t) = 2\phi_2(t) + \frac{\delta\sqrt{2}}{\alpha}\frac{D_{p+1}(z)}{D_p(z)} ,$$

where $z = \delta\sqrt{2}(t+\eta)$, $p = \beta_1/(\beta_2-\beta_1)$,

and D_p is the Weber parabolic cylinder function (cf. [7]).

$$D_p(z) = \Gamma(1/2)2^{p/2}\exp(-z^2/4)R_p(z) ,$$

with $R_p(z) = \dfrac{1}{\Gamma(1/2 - p/2)}\ {}_1F_1(-p/2,\ 1/2;\ z^2/2) - \dfrac{z\sqrt{2}}{\Gamma(-p/2)}\ {}_1F_1(1/2 - p/2,\ 3/2;\ z^2/2)$.

3. Proof of the Theorem. The demonstration of the conclusions is broken down into several stages.

LEMMA 1. *A function v is a solution of equation (1) on an interval I exactly when*

$$v = \phi_1 + \phi_2 - \frac{1}{\alpha}\frac{x'}{x}, \tag{2}$$

where $x(t) \neq 0$ *for t in I and is a solution of*

$$x''(t) - q(t)x(t) = 0 \tag{3}$$

with $$q = \alpha^2(\phi_1-\phi_2)^2 + \alpha(\beta_1+\beta_2).$$

Proof. We re-write equation (1) in the form

$$v' + 2Av + Bv^2 - C = 0$$

where $A = \alpha(\phi_1+\phi_2)$, $B = -\alpha$, and $C = 4\alpha\phi_1\phi_2$.

We then apply the result in Reid [5] that v is a solution of (1) on an interval I if, and only if, $v = u/x$, where $x(t) \neq 0$ on I and the pair (x,u) is a solution on I of the linear system

$$\begin{aligned} x' &= Ax + Bu \\ u' &= Cx - Au. \end{aligned} \tag{4}$$

But this system is equivalent to equation (3), as can be seen through the connection $u = (x'-Ax)/B$. Calculating $v = u/x$ then gives the form (2).

In order to transform equation (3) into more comprehensible forms, first we make a change of independent variable.

LEMMA 2. *The general solution of equation (1) is*

$$v(t) = \phi_1(t) + \phi_2(t) - \frac{\delta}{\alpha}\frac{w'(s)}{w(s)},$$

where $s = \delta(t+\eta)$, $\delta = \sqrt{\alpha(\beta_1-\beta_2)}$, $\eta = (\gamma_1-\gamma_2)/(\beta_1-\beta_2)$, *and* w *is a non-vanishing solution of the Weber equation*

$$w''(s) + (\epsilon - s^2)w(s) = 0 \tag{5}$$

with $\epsilon = (\beta_2+\beta_1)/(\beta_2-\beta_1)$.

While the form (5) is simpler than the form (3), it is not easy to see when its solutions are non-vanishing; the equation has an oscillatory interval about $s = 0$ if $\epsilon > 0$. However, we can make the change of dependent variable $y(s) = \exp(s^2/2)w(s)$, which transforms (5) into a Hermite equation and clearly preserves the non-vanishing of solutions. In fact, going through the calculations gives the following result.

LEMMA 3. *The general solution of equation* (1) *is*

$$v(t) = 2\phi_1(t) - \frac{\delta}{\alpha}\frac{y'(s)}{y(s)} , \tag{6}$$

where y *is a non-vanishing solution of the Hermite equation*

$$y''(s) - 2sy'(s) - 4ay(s) = 0 \tag{7}$$

with
$$a = \beta_1/2(\beta_1-\beta_2).$$

Before continuing, we point out that the procedure of transforming an equation of the form (3), where q is quadratic, first into the form (5) and then into the form (7) is well-known. It is used, for example, in solving the time-independent Schrödinger equation for a harmonic oscillator.

Now, the general solution of (7) can be expressed in terms of the confluent hypergeometric functions. In fact, we have the following result, which may be verified by direct calculation or by referring to Slater [7].

LEMMA 4. *Let* y_0 *and* y_1 *denote the solutions of equation* (7) *that satisfy the initial conditions*

$$y_0(0) = 0, \quad y_0'(0) = 1$$

$$y_1(0) = 1, \quad y_1'(0) = 0.$$

Then the functions y_0 *and* y_1 *are given by*

$$y_0(s) = s\ {}_1F_1(a + 1/2,\ 3/2;\ s^2), \tag{8}$$

$$y_1(s) = {}_1F_1(a,\ 1/2;\ s^2). \tag{9}$$

We now focus our attention on property (ii) in the statement of the Theorem. The next result shows that, up to a multiplicative constant, there is only one solution of (7) which, when substituted in (6), can possibly work.

LEMMA 5. *Every non-trivial solution* $y = c_0y_0 + c_1y_1$ *of equation* (7) *has the property that* $y'(s)/y(s) \to \infty$ *as* $s \to \infty$ *unless the constants* c_0 *and* c_1 *satisfy the relation*

$$c_0\Gamma(a) + 2c_1\Gamma(a + 1/2) = 0.$$

Proof. We apply two results about confluent hypergeometric functions given in Slater [7]. We have the derivative relation

$$\frac{d}{dz}\ {}_1F_1(a,\ b;\ z) = \frac{a}{b}\ {}_1F_1(a+1,\ b+1;\ z)$$

and the asymptotic expansion as $z \to \infty$

$${}_1F_1(a,\ b;\ z) = \frac{\Gamma(b)}{\Gamma(a)}\exp(z)\ z^{a-b}(1+O(z^{-1})).$$

If we now set $y = c_0y_0 + c_1y_1$ where $c_0^2 + c_1^2 > 0$ and apply these identities, then after some simplification we obtain the results that as $s \to \infty$,

$$y'(s) = \frac{\Gamma(1/2)\exp(s^2)s^{2a}}{\Gamma(a)\Gamma(a+1/2)}\Big\{\frac{\Gamma(a)}{2}c_0s^{-2}[1+O(s^{-2})] + c_0\Gamma(a)[1+O(s^{-2})] + 2c_1\Gamma(a+1/2)[1+O(s^{-2})]\Big\},$$

$$y(s) = \frac{\Gamma(1/2)\exp(s^2)s^{2a-1}}{2\Gamma(a)\Gamma(a+1/2)}\{c_0\Gamma(a)[1+O(s^{-2})] + 2c_1\Gamma(a+1/2)[1+O(s^{-2})]\} .$$

Therefore, as $s \to \infty$ we have $y'(s)/2sy(s) \to 1$, which implies $y'(s)/y(s) \to \infty$, unless the constants c_0 and c_1 make this form indeterminate. But this is precisely when $c_0\Gamma(a) + 2c_1\Gamma(a+1/2) = 0$.

Now recall that the variables s and t are related by $s = \delta(t+\eta)$, where $\delta > 0$, so the conditions $s \to \infty$ and $t \to \infty$ are equivalent. Then the form (6) for $v(t)$ shows that a solution that exists for large t will have $v(t) - 2\phi_1(t) \to -\infty$ as $t \to \infty$ unless c_0 and c_1 satisfy the relation stated in Lemma 5. Furthermore, the solution y of (7) enters into the form (6) only through the ratio y'/y, so one of the constants c_0 and c_1 may be chosen arbitrarily. Since we want to avoid solutions y that vanish, we choose $c_1 = 1$ and $c_0 = -2\Gamma(a+1/2)/\Gamma(a)$. We summarize what we have obtained so far as follows.

LEMMA 6. *A necessary condition for a solution v of equation (1) to have properties (i) and (ii) of the Theorem is that*

$$v(t) = 2\phi_1(t) - \frac{\delta}{\alpha}\frac{\psi'(s)}{\psi(s)} , \tag{10}$$

where $\psi(s) = \zeta y_0(s) + y_1(s)$ with $\zeta = -2\Gamma(a+1/2)/\Gamma(a)$ and y_0, y_1 are defined by formulas (8) and (9), respectively.

Notice that (10) is the form (A) of the solution v that is given in the remarks following the Theorem. We now proceed to show that the particular solution ψ of (7) forces the corresponding solution v of (1) to have properties (i) through (ix) of the Theorem.

LEMMA 7. *The solution $\psi = \zeta y_0 + y_1$ of equation (7) satisfies the inequalities $\psi(s) > 0$ and $\psi'(s) < 0$ for all s.*

Proof. The major theoretical tool we need to prove this result is stated in the Appendix. In order to apply that theorem to our problem, we put equation (7) in self-adjoint form by multiplying both sides by $\exp(-s^2)$. The result is the equivalent equation

$$(ry')' - py = 0,$$

where $r(s) = \exp(-s^2)$ and $p(s) = 4ar(s)$. Since r and p are continuous with $r(s) > 0$ and $p(s) > 0$ for all s, we can conclude that if $y = \theta y_0 + y_1$ is the solution of (7) with $\theta = \lim_{s\to\infty} - y_1(s)/y_0(s)$, then $y(s) > 0$ and $y'(s) < 0$ for all s. We now show that $\theta = \zeta$.

To do this, we proceed as in the proof of Lemma 5. Using the definition of y_0 and y_1, and the asymptotic expansion of the confluent hypergeometric functions again, we obtain, as $s \to \infty$:

$$y_1(s) = \frac{\Gamma(1/2)}{\Gamma(a)}\exp(s^2)s^{2a-1}(1+O(s^{-2}))$$

and $$y_0(s) = \frac{\Gamma(1/2)}{2\Gamma(a+1/2)}\exp(s^2)s^{2a-1}(1+O(s^{-2})).$$

This makes it clear that $\theta = -2\Gamma(a+1/2)/\Gamma(a) = \zeta$.

By referring to the form (10) and applying the result of Lemma 7, we immediately have;

COROLLARY. The solution v of equation (1) defined by (10) satisfies properties (iii) and (iv) and, a fortiori, satisfies properties (i) and (ii).

In order to tackle properties (v) through (viii), we develop the second form (B) of the solution v.

LEMMA 8. The solution $\psi = \zeta y_0 + y_1$ of equation (7) can be written in the form

(11) $$\psi(s) = \frac{2^a\Gamma(a+1/2)}{\Gamma(1/2)}\exp(z^2/4)D_p(z),$$

where $z = s\sqrt{2} = \delta\sqrt{2}(t+\eta)$ *and* D_p *is the Weber parabolic cylinder function with* $p = -2a = \beta_1/(\beta_2-\beta_1)$.

Proof. The result follows by using the definition of D_p and simplifying the right-hand side of (11).

LEMMA 9. *The solution* v *of equation* (1) *defined by* (10) *can be written in the form*

$$v(t) = 2\phi_2(t) + \frac{\delta\sqrt{2}}{\alpha}\frac{D_{p+1}(z)}{D_p(z)}. \tag{12}$$

Proof. To obtain this form, first we use formula (11) for ψ and calculate ψ'/ψ. Keeping in mind that $z = s\sqrt{2}$, we obtain

$$\frac{\psi'(s)}{\psi(s)} = s + \sqrt{2}\,\frac{D_p'(z)}{D_p(z)} .$$

Then we use the identity (cf. [4])

$$D_p'(z) = (z/2)D_p(z) - D_{p+1}(z). \tag{13}$$

The result is that

$$\frac{\psi'(s)}{\psi(s)} = 2s - \sqrt{2}\,\frac{D_{p+1}(z)}{D_p(z)} . \tag{14}$$

Substitution of this expression into formula (10) and use of the relation $s = \delta(t+\eta)$ give the form (12) for v, which is the form (B) that was claimed.

Properties (v), (vi), and (vii) can now be attacked by using the following asymptotic expansion of the parabolic cylinder functions (cf. [4]). As $z \to \infty$,

$$D_p(z) = \exp(-z^2/4)\, z^p[1 - \frac{p(p-1)}{2z^2} + O(z^{-4})]. \tag{15}$$

LEMMA 10. *If* $\psi = \zeta y_0 + y_1$, *then* $\psi(s) \to 0^+$ *as* $s \to \infty$.

Proof. We return to formula (11) for ψ, keeping in mind the connection

$z = s\sqrt{2}$ and the fact that $p = -2a < 0$. Substitution of the result of (15) into (11) yields, as $s \to \infty$,

$$(16) \qquad \psi(s) = \frac{2^a\Gamma(a+1/2)}{\Gamma(1/2)} z^p[1 - \frac{p(p-1)}{2z^2} + O(z^{-4})],$$

so $\psi(s) \to 0$ as $s \to \infty$. Since $\psi(s) > 0$ for all s, the conclusion follows.

COROLLARY. *The solution v of equation* (1) *defined by* (10) *satisfies property* (v).

Proof. Using the form (10) with the connection $s = \delta(t+\eta)$ shows that if we fix some t_0 and let s_0 be the corresponding value of s, then

$$(17) \qquad \int_{t_0}^{t} (v-2\phi_1) = -\frac{1}{\alpha}\int_{s_0}^{s} (\psi'/\psi) = -\frac{1}{\alpha}\ln[\psi(s)/\psi(s_0)].$$

But $s \to \infty$ as $t \to \infty$, so $\int_{t_0}^{\infty} (v-2\phi_1) = \infty$ follows immediately from Lemma 10.

For the second integral, we again fix t_0 and then apply property (iv) to an interval $[t_0,t]$. The result is

$$\int_{t_0}^{t} (v-2\phi_2) > \int_{t_0}^{t} (2\phi_1-2\phi_2)$$

$$= (\beta_1-\beta_2)(t^2-t_0^2) + (\gamma_1-\gamma_2)(t-t_0)$$

which $\to \infty$ as $t \to \infty$, since $\beta_1 > \beta_2$.

LEMMA 11. *If* $\psi = \zeta y_0 + y_1$, *then* $\psi'(s)/\psi(s) \to 0$ *as* $s \to \infty$.

Proof. We begin by using formula (14), recalling again that $z = s\sqrt{2}$. The result is

$$\frac{\psi'(s)}{\psi(s)} = \sqrt{2}[z - \frac{D_{p+1}(z)}{D_p(z)}].$$

But $z \to \infty$ as $s \to \infty$, and if we use just the result

$$D_p(z) = \exp(-z^2/4)\, z^p(1 + O(z^{-2}))$$

from formula (15), we obtain

$$z - \frac{D_{p+1}(z)}{D_p(z)} = z - \frac{z^{p+1}}{z^p}\,\frac{[1 + O(z^{-2})]}{[1 + O(z^{-2})]} = \frac{z\,O(z^{-2})}{1 + O(z^{-2})}\ ,$$

which approaches zero as $z \to \infty$.

COROLLARY. *The solution* v *of equation* (1) *defined by* (10) *satisfies property* (vi).

The asymptotic expansion (15) can be used again to establish property (vii). First, we isolate the most important calculation that is involved.

LEMMA 12. *The following limit relation holds for the parabolic cylinder functions*:

$$\left(\frac{D_{p+1}}{D_p}\right)'(z) \to 1 \quad \textit{as} \quad z \to \infty.$$

Proof. *After using the quotient rule to calculate the indicated derivative, we use in turn the identity* (13) and its companion (cf. [4])

$$D'_{p+1}(z) = (p+1)D_p(z) - (z/2)D_{p+1}(z).$$

The result is

$$\left(\frac{D_{p+1}}{D_p}\right)'(z) = (p+1) + \left(\frac{D_{p+1}}{D_p}\right)^2(z) - z\left(\frac{D_{p+1}}{D_p}\right)(z). \tag{18}$$

If we apply (15) and do some re-shuffling of factors, we find that

$$\left(\frac{D_{p+1}}{D_p}\right)^2(z) - z\left(\frac{D_{p+1}}{D_p}\right)(z) = \frac{[1-p(p+1)/2z^2+O(z^{-4})]}{[1-p(p-1)/2z^2+O(z^{-4})]}\ [-p + z\,O(z^{-4})],$$

which approaches $\underline{-p}$ as $z \to \infty$. The conclusion then follows immediately.

COROLLARY. *The solution* v *of equation* (1) *defined by* (10) *satisfies property* (vii).

Proof. If we look at the form (12) of the solution, we obtain

$$v'(t) = 2\phi_2'(t) + \frac{\delta\sqrt{2}}{\alpha}\left(\frac{D_{p+1}}{D_p}\right)'(z)\,\frac{dz}{dt}$$

$$= 2\beta_2 + 2\,\frac{\delta^2}{\alpha}\left(\frac{D_{p+1}}{D_p}\right)'(z),$$

since $z = \delta\sqrt{2}(t+\eta)$. But $z \to \infty$ as $t \to \infty$, so Lemma 12 implies that

$$v'(t) \to 2\beta_2 + 2\,\frac{\delta^2}{\alpha} \quad \text{as} \quad t \to \infty.$$

Using the definition $\delta = \sqrt{\alpha(\beta_1-\beta_2)}$ then gives the result.

Next, we use the following result about the parabolic cylinder functions.

LEMMA 13. _If_ $\beta_2 < 0 < \beta_1$ _and_ $p = \beta_1/(\beta_2-\beta_1)$, _then_:

(i) $D_p(z) > 0$ _for all_ z;

(ii) D_{p+1} _has exactly one real zero_ z_0, _and_ $D_{p+1}(z) > 0$ _exactly when_ $z > z_0$.

Proof. The hypotheses imply that $0 < p+1 < 1$. Hence, the result follows immediately (cf. [1]).

COROLLARY. _The solution_ v _of equation_ (1) _defined by_ (10) _satisfies property_ (viii).

Proof. If we apply Lemma 13 to the form (12) of the solution, we see that

$$v(t) - 2\phi_2(t) > 0 \quad \text{for} \quad t > t_0$$

and

$$v(t_0) - 2\phi_2(t_0) = 0$$

exactly when t_0 satisfies $z_0 = \delta\sqrt{2}(t_0+\eta)$. A simple calculation using the definitions of ϕ_1, ϕ_2, δ and η then gives the result.

LEMMA 14. *The functions* $\bar{\Phi}_i$ *defined in* (ix) *are related by*

$$\bar{\Phi}_2(t) = G(t)\,\bar{\Phi}_1(t), \tag{19}$$

where $G(t) = \exp(s_0^2 - s^2)$ *with* $s = \delta(t+\eta)$ *and* $s_0 = \delta(t_0+\eta)$.

Proof. Since $x_i(t) = \alpha[v(t)-2\phi_i(t)]$, we have $x_2(t) = x_1(t) + 2\delta^2(t+\eta)$, from which (19) follows easily.

COROLLARY. *The functions* $\bar{\Phi}_i$ *satisfy* (ixa).

Proof. Since t_0 satisfies (viii), we have

$$(\beta_1-\beta_2)(t_0+\eta) = \phi_1(t_0) - \phi_2(t_0) = z_0\sqrt{(\beta_1-\beta_2)/2\alpha},$$

which is negative because the zero z_0 of $D_{p+1}(z)$ with $0 < p+1 < 1$ is negative (cf. [1]). Hence, $t_0+\eta < 0$ since $\beta_1 > \beta_2$, so $s_0 < 0$. Therefore, $G(t) > 1$ exactly when $s_0 < s < -s_0$, i.e., when $t_0 < t < -t_0 - 2\eta$.

LEMMA 15. *The solution* $\psi = \zeta y_0 + y_1$ *of equation* (7) *has* $\psi''(s) > 0$ *for all* s.

Proof. If there were a value s_1 at which $\psi''(s_1) = 0$, then by taking the derivative with respect to s of both sides of (7) with y replaced by ψ, we would have $\psi'''(s_1) = (2+4a)\psi'(s_1)$. Since just the hypotheses $\beta_2 < \beta_1$ and $\beta_1 > 0$ imply $a > 0$, and Lemma 7 implies $\psi'(s_1) < 0$, it follows that $\psi'''(s_1) < 0$. This says that at s_1, ψ changes from being convex to concave, and hence, by this very argument, can never change back to being convex. But that contradicts the result of Lemma 7. So, $\psi''(s)$ is never zero. But, by replacing s by 0 in (7), we have $\psi''(0) = 4a > 0$. Since ψ'' is continuous, the result follows.

COROLLARY. *The function* f_1 *satisfies* (ixb).

Proof. Since property (iv) implies $x_1(t) > 0$ for all t, it follows from the definition of f_1 that $f_1(t) > 0$ for all t. To show that f_1 is decreasing, notice that form (A) of the solution to equation (1) implies that $x_1(t) = -\delta\psi'(s)/\psi(s) = \frac{-d}{dt}\psi(s)$.

Hence, we have

$$\bar{\Phi}_1(t) = \psi(s)/\psi(s_0), \tag{20}$$

where, as in Lemma 14, $s = \delta(t+\eta)$ and $s_0 = \delta(t_0+\eta)$. But then $f_1(t) = -\bar{\Phi}_1'(t) = -\delta\psi'(s)/\psi(s_0)$, so $f_1'(t) = -\delta^2\psi''(s)/\psi(s_0) < 0$ by Lemmas 7 and 15.

LEMMA 16. The function f_2 satisfies (ixc).

Proof. That $f_2(t) > 0$ for $t > t_0$ and $f_2(t_0) = 0$ follows from the definition of f_2 and (viii). Next, we show that $f_2(t) \to 0$ as $t \to \infty$. For, the definition of f_2 and relation (19) imply $f_2 = -\bar{\Phi}_2' = -(G\bar{\Phi}_1)'$. Using (20) then gives

$$f_2(t) = \delta G(t)\bar{\Phi}_1(t)[2s - \psi'(s)/\psi(s)]. \tag{21}$$

Since $t > t_0$ corresponds to $s > s_0$, (20) and Lemma 7 imply

$$0 < \bar{\Phi}_1(t) < 1 \quad \text{for} \quad t > t_0. \tag{22}$$

Also, $\psi'(s)/\psi(s) \to 0$ as $t \to \infty$ by Lemma 11. Finally, $2sG(t) \to 0$ as $t \to \infty$ by the definition of G. Hence, f_2 is a continuous function with $f_2(t) > 0$ on (t_0,∞) while $f_2(t_0) = 0 = f_2(\infty)$, so f_2 has an absolute maximum in (t_0,∞). To facilitate the calculation of f_2', we first use (14) and (19) to re-write (21) as

$$f_2(t) = \delta\,\bar{\Phi}_2(t)\, D_{p+1}(z)/D_p(z). \tag{23}$$

with $z = \sqrt{2}\delta(t+\eta)$ as before. Taking the derivative of both sides of (23) with respect to t and using (18) as well as $\bar{\Phi}_2' = -f_2$, we find that

$$f_2'(t) = 2\delta^2\bar{\Phi}_2(t)[p+1 - zD_{p+1}(z)/D_p(z)].$$

Since $t > t_0$ corresponds to $z > z_0$, where $\underline{z_0}$ is the (negative) zero of $D_{p+1}(z)$, it follows from Lemma 13 that $f_2'(t) > 0$ for $z_0 < z \leq 0$, i.e., $t_0 < t \leq -\eta$. Hence, the maximum of $\underline{f_2}$ occurs at some $t_1 > -\eta$.

In order to deal with (ixd) and (ixe), we use $f_i = -\bar{\Phi}_i'$ and integration by parts to get

$$\int_{t_0}^{t} \tau f_i(\tau)d\tau = t_0 - t\,\bar{\Phi}_i(t) + \int_{t_0}^{t} \bar{\Phi}_i(\tau)d\tau. \tag{24}$$

Then (19) and (22) easily yield

LEMMA 17. *The function* f_2 *satisfies* (ixd).

Finally, (20) and the asymptotic expansion expansion (16) for ψ show that $\bar{\Phi}_1(t)$ behaves like t^p as $t \to \infty$, where $p = \beta_1/(\beta_2-\beta_1)$. Since $\beta_2 < 0 < \beta_1$ implies that $-1 < p < 0$, it follows readily from (24) that

LEMMA 18. *The function* f_1 *satisfies* (ixe).

4. An application to a noisy duel problem. In [3] and [6] appears a dynamic programming approach to the $\underline{m}$ vs. $\underline{n}$ equal-accuracy noisy duel problem, where the positive integers $m < n$ represent the units of ammunition the two players have. The approach begins by allowing either of the players to fire a unit of ammunition only at times corresponding to points of a discrete grid of the interval [0,1], which is interpreted as the interval of probabilities of either player destroying the other if a unit is fired. This produces a finite sequence of simultaneous games whose 2×2 pay-off matrices are determined by proceeding backwards inductively from the game where the probability of destruction is unity.

Attention is focussed on an interval of grid points at which the players have no pure strategy and which surrounds the critical probability $1/(m+n)$. It is found, under suitable hypotheses suggested by computer implementation of the above approach, that the value of the game in this interval of grid points satisfies a difference equation. Dividing both sides of this equation by an appropriate normalization factor and letting the mesh of the grid on [0,1] approach zero leads to a normalized value $v = v_{m,n}$ of the game that satisfies a Riccati equation of the form (1) on the interval $[-1,\infty)$, with

$$\alpha = (m+n)^2(m+n-2)/(n-m),$$

$$\beta_1 = m\,c_{m,n}^2/(m+n-1) \quad , \quad \beta_2 = (-n/m)\beta_1,$$

$$\gamma_1 = \frac{(m+n-1)}{2(m+n)} \cdot \frac{c_{m,n}}{c_{m,n-1}} \cdot v_{m,n-1}(-1) \quad \text{for} \quad 1 \leq m < n-1,$$

while $\gamma_1 = 0$ for $m = n - 1$,

$$\gamma_2 = \frac{(m+n-1)}{2(m+n)} \cdot \frac{c_{m,n}}{c_{m-1,n}} \cdot v_{m-1,n}(-1) \quad \text{for} \quad 1 < m \leq n-1,$$

while $\gamma_2 = 0$ for $m = 1$.

Here, it is known that the constants $c_{i,j}$ are positive for $i < j$, but analytic expressions for these constants are not known *a priori*. However, the hypotheses $\alpha > 0$ and $\beta_2 < 0 < \beta_1$ are evidently satisfied. Also, it is established in [3] and [6] that the initial condition $v(t_0) - 2\phi_2(t_0) = 0$ is to hold when $t_0 = -1$. But this does not seem to be enough information to attack the existence and uniqueness problem for (1) on $[-1,\infty)$.

Instead, attention is turned to the functions defined in (ix) with $t_0 = -1$, which corresponds in the normalization process to the beginning of the interval surrounding probability $1/(m+n)$ in which random strategies are to be employed. The functions $\bar{\Phi}_1(t)$ and $\bar{\Phi}_2(t)$ represent, respectively, the probability that the weaker player and the stronger player has a normalized time of first fire occurring at or after $\underline{t}$. The functions $x_i(t)$ represent the corresponding hazard rates for the cdf's $\Phi_i(t) \equiv 1-\bar{\Phi}_i(t)$ and the functions $f_i(t)$ represent their densities.

One of the facts derived in [3] and [6] is that the weaker player's hazard rate $x_1(t)$ is to be positive for $t \geq -1$. Somewhat surprisingly, the assumption that the solution of (1) exists for large $\underline{t}$ and that $x_1(t)$ be positive for large $\underline{t}$ produces not only the global existence and uniqueness result for (1) proved herein, but also some properties of the complements $\bar{\Phi}_i(t)$ of the cdf's $\Phi_i(t)$ that could not be surmised by studying the computer runs for the 2×2 games, namely:

Property (v) implies that

$$\int_{-1}^{\infty} x_i(t)dt = \infty \quad \text{for} \quad i = 1,2,$$

so that
$$\lim_{t\to\infty} \bar{\Phi}_i(t) = \exp\left(-\int_{-1}^{\infty} x_i(\tau)d\tau\right) = 0,$$

which implies
$$\lim_{t\to\infty} \Phi_i(t) = 1.$$

This says that the probability is unity that each player fires at some time in the normalized interval $-1 \leq t < \infty$ during which random strategies are employed.

Property (ixa) states that the probability is greater not only for the weaker player firing before the stronger one for normalized times near $t_0 = -1$, but also for the weaker player firing after the stronger one for large $\underline{t}$.

Properties (ixb) and (ixc) combine to show that the mode of $\underline{f_1}$ occurs at $t_0 = -1$ while that of $\underline{f_2}$ occurs at a value of $\underline{t}$ greater than that at which the two players' hazard rates are equal, which is in turn greater than -1.

Properties (ixd) and (ixe) state that the expectation of Φ_2 is finite while that of Φ_1 is infinite.

Finally, information about the constants $c_{i,j}$ and the initial values $v_{i,j}(-1)$ for $i < j$ can be obtained by means of a complicated recursive process. First, the above-mentioned fact that $v_{m,n}(-1)-2\phi_2(-1) = 0$ yields, from the formulas for α, β_1, β_2, γ_1, and γ_2:

$$v_{m,n}(-1) = \frac{2n}{(m+n-1)} \cdot c_{m,n}^2 + \frac{(m+n-1)}{(m+n)} \cdot \frac{c_{m,n}}{c_{m-1,n}} \cdot v_{m-1,n}(-1) \quad \text{if} \quad 1 < m \leq n-1$$

and

$$v_{1,n}(-1) = 2c_{1,n}^2.$$

Then, this relation coupled with the fact from [3] and [6] that $v_{m,n}(t) - 2\phi_2(t) > 0$ for $t > -1$ implies, by property (viii), that

$$\phi_1(-1) - \phi_2(-1) = z_{m,n}\sqrt{(\beta_1-\beta_2)/2\alpha},$$

where $z_{m,n}$ is the zero of the Weber parabolic cylinder function D_{p+1}, with $p = -m/(m+n)$. Solving this relation for $c_{m,n}$, using the fact that $c_{m,n} > 0$, gives

$$c_{m,n} = -z_{m,n}A_{m,n} + B_{m,n}$$

where

$$A_{m,n} = [(n-m)(m+n-1)/2(m+n)^3(m+n-2)]^{1/2}$$

and

$$B_{m,n} = \frac{(m+n-1)^2}{2(m+n)^2}\left[\frac{v_{m,n-1}(-1)}{c_{m,n-1}} - \frac{v_{m-1,n}(-1)}{c_{m-1,n}}\right] \quad \text{for} \quad 1 < m < n-1$$

while for $n > 2$,

$$B_{1,n} = \frac{n}{2(n+1)} \cdot \frac{v_{1,n-1}(-1)}{c_{1,n-1}} = \frac{n}{(n+1)}\, c_{1,n-1}$$

and

$$B_{n-1,n} = -\frac{(n-1)}{2n-1} \cdot \frac{v_{n-2,n}(-1)}{c_{n-2,n}},$$

and lastly that

$$B_{1,2} = 0.$$

Thus, we have $c_{1,2} = -z_{1,2}/\sqrt{27}$ at the beginning of the recursive chain.

Next, we find that
$$c_{1,n} = -z_{1,n}A_{1,n} + B_{1,n}$$
expresses $c_{1,n}$ in terms of $c_{1,n-1}$ for $n > 2$. Hence, the formula $v_{1,n}(-1) = 2c_{1,n}^2$ determines these initial values in a simple recursive way. It is clear then that the values $c_{m,n}$ and $v_{m,n}(-1)$ can eventually be calculated in terms of m,n and the zeroes $z_{m,n}$, but simple formulas for those values are not apparent.

Thus, it is indeed fortunate that the analysis presented here that is germane to the noisy duel problem does not depend on specific information about the coefficients in (1) beyond the hypotheses of the Theorem. That lack of information is compensated for by the condition that properties (i) and (ii) are to hold.

Appendix. The proof of Lemma 7 depends on the following results, which can be derived by straightforward modifications of the argument given in section 9.2 of Hille [2].

THEOREM. *Suppose r and p are continuous functions such that* $r(s) > 0$ *and* $p(s) \geq 0$ *for s real. Let y_0 and y_1 be the solutions of*

(A1) $$(ry')' - py = 0$$

that satisfy the initial conditions

$$y_0(0) = 0, \quad y_0'(0) = 1$$

and $$y_1(0) = 1, \quad y_1'(0) = 0.$$

Then: (a) *The limits*

$$\theta = \lim_{s\to\infty} - \frac{y_1(s)}{y_0(s)}$$

$$\mu = \lim_{s\to\infty} - \frac{y_1'(s)}{y_0'(s)}$$

exist, and $\theta \leq \mu$.

(b) *The solutions of* (A1) *passing through the point* (0,1) *that have* $y(s) > 0$ *and* $y'(s) \leq 0$ *for* $s > 0$ *are precisely those solutions* $y = \lambda y_0 + y_1$ *that have* $\theta \leq \lambda \leq \mu$. *Moreover, every such solution satisfies*:

(b_1) $y(s) > 0$ *and* $y'(s) \leq 0$ *for all real s.*

(b_2) $y'(s) < 0$ *over any interval on which* $p(s)$ *does not vanish identically.*

(c) $\theta = \mu$ *exactly when*

$$\int_0^\infty [r(y_0')^2 + p(y_0)^2] = \infty,$$

a sufficient condition for which is the divergence of $\int_0^\infty \frac{1}{r}$.

References

[1] A. ERDELYI, W. MAGNUS, F. OBERHETTINGER, and F. G. TRICOMI, Higher Transcendental Functions, Volume II, McGraw-Hill, New York 1953.

[2] EINAR HILLE, Lectures on Ordinary Differential Equations, Addison-Wesley, Reading, Mass., 1969.

[3] J. COLBY KEGLEY, An Asymptotic Stochastic View of a Noisy Duel (II), Preprint.

[4] I. YE. KIREYEVA and K. A. KARPOV, Tables of Weber Functions, Volume I, Pergamon Press, New York, 1961.

[5] W. T. REID, Riccati Differential Equations, Mathematics in Science and Engineering, Vol. 86, Academic Press, New York, 1972.

[6] DAN R. ROYALTY, J. COLBY KE~~B~~GLEY, H. T. DAVID, and R. W. BERGER, An Asymptotic Stochastic View of a Noisy Duel (I), Preprint.

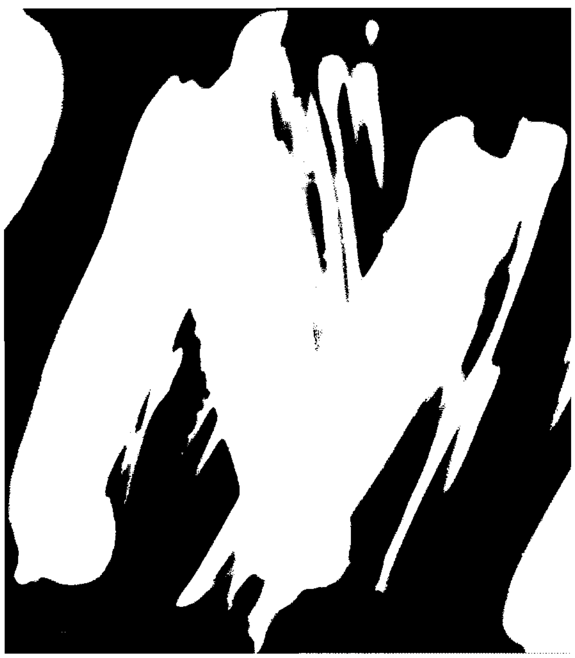
[7] LUCY SLATER, Confluent Hypergeometric Functions, Cambridge University Press, 1960.

END

DATE

FILMED

-82



AD-A116 021 IOWA STATE UNIV AMES DEPT OF STATISTICS F/G 12/1
A GLOBAL EXISTENCE AND UNIQUENESS THEOREM FOR A RICCATI EQUATIO--ETC(U)
1981 J C KEGLEY AFOSR-78-3518
UNCLASSIFIED AFOSR-TR-82-0491 NL

2 of 2

END
DATE FILMED
8-82
DTIC

DA
116

SUPPLEMENTAR

INFORMATION

AD-A116021

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER AFOSR-TR- 82-0491

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle) A GLOBAL EXISTENCE AND UNIQUENESS THEOREM FOR A RICCATI EQUATION

5. TYPE OF REPORT & PERIOD COVERED ~~Final~~ Interim 1 DEC 77-31 AUG 81

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s) J. Colby Kegley

8. CONTRACT OR GRANT NUMBER(s) AFOSR-78-3518

9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Statistics, Iowa State University, Ames IA 50011

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS PE61102F; 2304/A5

11. CONTROLLING OFFICE NAME AND ADDRESS Mathematical & Information Sciences Directorate, Air Force Office of Scientific Research, Bolling AFB DC 20332

12. REPORT DATE 1981

13. NUMBER OF PAGES 21

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

15. SECURITY CLASS. (of this report) UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Riccati differential equation.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

It is proved that a Riccati differential equation of a particular form has a unique solution satisfying the conditions that it is to exist for large values of the independent variable t and to have its graph stay above a certain line for large t. It is then proved that the solution exists for all t. Two forms of the solution are developed in terms of the confluent hypergeometric functions. An application of these results is made to an asymptotic stochastic analysis of a noisy duel problem.

DD FORM 1473, 1 JAN 73 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

ADA116021